Six radical–trigonometric closed forms for $\cos\frac{2\pi}{p}$

This short note supplies "nontrivial" radical-trigonometric expressions for $\cos(2\pi/7)$, $\cos(2\pi/11)$, $\cos(2\pi/13)$, $\cos(2\pi/19)$, $\cos(2\pi/23)$, and $\cos(2\pi/29)$.

$p = 7$

The minimal polynomial of $\cos\frac{2\pi}{7}$ is $-1 - 4x + 4x^2 + 8x^3$. It is given by this expression:

$$\cos\frac{2\pi}{7} = \frac{1}{6}\left(2\sqrt{7}\cos\left(\frac{1}{3}\arctan\left(3\sqrt{3}\right)\right) - 1\right)$$

$p = 11$

The minimal polynomial of $\cos\frac{2\pi}{11}$ is $1+6x-12x^2-32x^3+16x^4+32x^5$. It is given by this expression:

$$\cos\frac{2\pi}{11} = \frac{1}{10}\left(2\sqrt{11}\left(\cos\left(\frac{1}{5}\left(\arctan\left(\frac{5}{\sqrt{877+\frac{1958}{\sqrt{5}}}}\right)+\pi\right)\right)\right.\right.$$
$$\left.\left.+\cos\left(\frac{1}{5}\left(\arctan\left(\frac{5}{\sqrt{877-\frac{1958}{\sqrt{5}}}}\right)+\pi\right)\right)\right) - 1\right)$$

$p = 13$

The minimal polynomial of $\cos\frac{2\pi}{13}$ is $-1 + 6x + 24x^2 - 32x^3 - 80x^4 + 32x^5 + 64x^6$. It is given by this expression:

$$\cos\frac{2\pi}{13} = \frac{1}{12}\left(2\sqrt{26-2\sqrt{13}}\cos\left(\frac{1}{3}\arctan\left(\frac{5+2\sqrt{13}}{3\sqrt{3}}\right)\right) + \sqrt{13} - 1\right)$$

$p = 19$

The minimal polynomial of $\cos\frac{2\pi}{19}$ is $1 + 10x - 40x^2 - 160x^3 + 240x^4 + 672x^5 - 448x^6 - 1024x^7 + 256x^8 + 512x^9$. It is given by this expression:

$$\cos\frac{2\pi}{19} = \frac{1}{18}\left(2\sqrt{19}\left(\sqrt{3\left(1+\frac{4}{\sqrt{19}}\cos\left(\frac{1}{3}\arctan\left(\frac{3\sqrt{3}}{7}\right)\right)\right)}\right.\right.$$
$$\times\cos\left(\frac{1}{3}\arctan\left(\frac{38\sin\left(\frac{1}{3}\arctan\left(\frac{21\sqrt{3}}{11}\right)\right)+\sqrt{2641}\sin\left(\frac{1}{3}\arctan\left(\frac{7959\sqrt{3}}{3751}\right)\right)}{38+38\cos\left(\frac{1}{3}\arctan\left(\frac{21\sqrt{3}}{11}\right)\right)+\sqrt{2641}\cos\left(\frac{1}{3}\arctan\left(\frac{7959\sqrt{3}}{3751}\right)\right)}\right)\right)$$
$$\left.\left.-\cos\left(\frac{1}{3}\left(\arctan\left(\frac{3\sqrt{3}}{7}\right)+\pi\right)\right)\right)-1\right)$$

$p = 23$

The minimal polynomial of $\cos\frac{2\pi}{23}$ is $-1-12x+60x^2+280x^3-560x^4-1792x^5+1792x^6+4608x^7-2304x^8-5120x^9+1024x^{10}+2048x^{11}$.

Expression. Write $c_k := \cos\frac{2\pi k}{11}$ for $k = 1, \ldots, 5$, obtained from the inner (degree-5) layer

$$2\cos\frac{2\pi m_j}{11} = -\frac{1}{5} + \frac{2\sqrt{11}}{5}\left[\cos\left(g_1 - \frac{2\pi j}{5}\right) + \cos\left(g_2 - \frac{4\pi j}{5}\right)\right], \quad (m_0, \ldots, m_4) = (1, 2, 4, 3, 5),$$

where, with $x_1 = -\frac{979+275\sqrt{5}}{4\times 11^{5/2}}$, $x_2 = \frac{-979+275\sqrt{5}}{4\times 11^{5/2}}$,

$$g_1 = \frac{2\pi - 2\arctan\sqrt{\frac{1-x_1}{1+x_1}}}{5}, \qquad g_2 = \frac{2\pi - 2\arctan\sqrt{\frac{1-x_2}{1+x_2}}}{5}.$$

Set

$$\begin{aligned}
\Re A_1 &= 10142747 - 13094774\,c_1 + 21879187\,c_2 + 23715967\,c_3 + 9094085\,c_4 + 6914996\,c_5\\
\Re A_2 &= 10142747 + 6914996\,c_1 - 13094774\,c_2 + 9094085\,c_3 + 21879187\,c_4 + 23715967\,c_5\\
\Re A_3 &= 10142747 + 9094085\,c_1 + 23715967\,c_2 - 13094774\,c_3 + 6914996\,c_4 + 21879187\,c_5\\
\Re A_4 &= 10142747 + 23715967\,c_1 + 6914996\,c_2 + 21879187\,c_3 - 13094774\,c_4 + 9094085\,c_5\\
\Re A_5 &= 10142747 + 21879187\,c_1 + 9094085\,c_2 + 6914996\,c_3 + 23715967\,c_4 - 13094774\,c_5
\end{aligned}$$

and, with $X_t = \frac{\Re A_t}{23^{11/2}}$,

$$\theta_t = \frac{-2\arctan\sqrt{\frac{1-X_t}{1+X_t}} + 2\pi N_t}{11}, \quad (N_1, \ldots, N_5) = (0, 3, 3, 1, -2).$$

$\cos\frac{2\pi}{23}$ is given by this expression:

$$\cos\frac{2\pi}{23} = \frac{1}{22}\left(2\sqrt{23}(\cos\theta_1 + \cos\theta_2 + \cos\theta_3 + \cos\theta_4 + \cos\theta_5) - 1\right)$$

$p = 29$

The minimal polynomial of $\cos\frac{2\pi}{29}$ is $-1 + 14x + 112x^2 - 448x^3 - 2016x^4 + 4032x^5 + 13440x^6 - 15360x^7 - 42240x^8 + 28160x^9 + 67584x^{10} - 24576x^{11} - 53248x^{12} + 8192x^{13} + 16384x^{14}$.

Expression. Write $d_k := \cos\frac{2\pi k}{7}$ for $k = 1, 2$, which satisfies

$$2\cos\frac{2\pi m_j}{7} = -\frac{1}{3} + \frac{2\sqrt{7}}{3}\left[\cos\left(\frac{1}{3}\arctan\left(3\sqrt{3}\right) - \frac{2\pi j}{3}\right)\right], \quad (m_0, m_1) = (1, 2).$$

Set

$$\begin{aligned}
\rho_1 &= 2\Big(29 - \sqrt{29}\,(3 + 4d_1 + 8d_2)\Big) \\
\rho_2 &= 2\Big(29 + \sqrt{29}\,(1 + 8d_1 + 4d_2)\Big) \\
\rho_3 &= 2\Big(29 - \sqrt{29}\,(1 + 4d_1 - 4d_2)\Big)
\end{aligned}$$

and

$$\begin{aligned}
\Re A_1 &= -13253 + 294756\,d_1 + 147784\,d_2 - \sqrt{29}\,(5983 + 16632\,d_1 - 41160\,d_2) \\
\Re A_2 &= -87145 - 147784\,d_1 + 146972\,d_2 - \sqrt{29}\,(26563 + 41160\,d_1 + 57792\,d_2) \\
\Re A_3 &= -160631 - 146972\,d_1 - 294756\,d_2 + \sqrt{29}\,(2333 + 57792\,d_1 + 16632\,d_2).
\end{aligned}$$

With $X_a = \frac{\Re A_a}{4\times\rho_a^{7/2}}$, define

$$\theta_a = \frac{-2\arctan\sqrt{\frac{1-X_a}{1+X_a}} + 2\pi N_a}{7}, \quad (N_1, N_2, N_3) = (0, 0, 3).$$

$\cos\frac{2\pi}{29}$ is given by this expression:

$$\cos\frac{2\pi}{29} = \frac{1}{28}\Big(2(\sqrt{\rho_1}\,\cos\theta_1 + \sqrt{\rho_2}\,\cos\theta_2 + \sqrt{\rho_3}\,\cos\theta_3) + \sqrt{29} - 1\Big)$$